Cohesive Powers of Decomposable Structures

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December 6th, 2022

Introduction

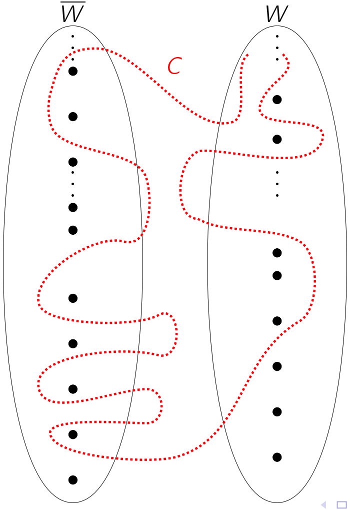
Overview

In this talk, we will define the cohesive power construction and see what it does to a few easily defined classes of structures. Introduced by Dimitrov, cohesive powers are

- inspired by constructions used to build particular countable models of PA.
- a way of adding "limit points" to a computable structure. Sieving through a cohesive set allows us to filter out contradictory behavior.
- analogues of ultrapowers that use a cohesive set instead of an ultrafilter.

Cohesive Sets

We call an infinite set $C \subseteq \omega$ cohesive if, for each c.e. set W, we have that either $C \subseteq^* W$ or $C \subseteq^* \overline{W}$.



Existence of Cohesive Sets

To construct a cohesive set C, fix enumeration $W_1, W_2, \ldots$ of the c.e. sets. First, we will construct infinite sets $A_0, A_1, \ldots$ such that for each $e < i$, the set A_i is cohesive for W_e, i.e. that either $A_i \subseteq^* W_e$ or $A_i \subseteq^* \overline{W}_e$.

- Let $A_0 = \omega$. To construct A_{i+1}, check whether one of $A_i \cap W_{i+1}$ or $A_i \cap \overline{W}_{i+1}$ is finite.
  - If so, then A_i is cohesive for W_{i+1}. let $A_{i+1} = A_i$.
  - If not, then both $A_i \cap W_{i+1}$ and $A_i \cap \overline{W}_{i+1}$ are infinite.
    - Let $A_{i+1} = A_i \cap W_{i+1}$.

Let $C = \{c_0 < c_1 < c_2 < \ldots\}$ such that $c_i \in A_i$. Then, $C \subseteq^* A_i$ for each i, so C is cohesive.

Essential Property of Cohesive Sets

Let C be a cohesive set. Let S be any set such that $C \subseteq^* S$. Let $A_1, \ldots, A_n$ be a disjoint family of c.e. or co-c.e. sets such that $\bigcup_{i \leq n} A_i =^* S$. Then, $C \subseteq^* A_i$ for some $i \leq n$.

Proof.

Suppose that $C \subseteq^* \overline{A_i}$ for each $i \leq n$. Then, $C \subseteq^* \bigcap_{i \leq n} \overline{A_i} =^* \overline{S}$, contradicting that $C \subseteq^* S$. □

Example: Adding limit points to $(\omega, \leq_\omega)$

Fix language $\mathcal{L} = \{\leq\}$ and $\mathcal{L}$-structure $(\omega, \leq_\omega)$. Suppose that we want to define an $\mathcal{L}$-structure whose elements are sequences of elements of ω. We try interpreting $\leq$ pointwise.

Example

Let $\alpha = 1, 2, 3, 4, 5, \ldots$ and let $\beta = 0, 1, 2, 3, 4, 5, \ldots$. Then, since $\alpha(i) \leq_\omega \beta(i)$ for all $i \in \omega$, we should set $\alpha \leq \beta$.

But what if there is contradictory behavior?

Example

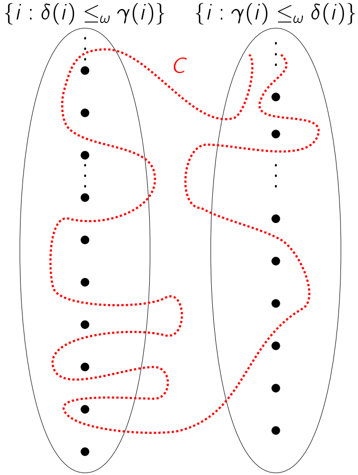
Let $\gamma = 1, 2, 3, 4, 5, \ldots$ and let

$$\delta(i) = \begin{cases} \gamma(i) - 1 & \text{if } i \text{ is odd} \\ \gamma(i) + 1 & \text{if } i \text{ is even} \end{cases}$$

It is not clear whether we should interpret $\gamma \leq \delta$ or $\delta \leq \gamma$.

Deciding Ambiguity

We can let a cohesive set C decide.



C says that $\gamma \leq \delta$.

Universe of Cohesive Powers

Cohesive sets only decide c.e. sets. Thus, we restrict our universe to partial computable functions. Fix cohesive set C.

- Let D be the set of partial computable functions φ such that $C \subseteq^* \mathrm{dom}(\varphi)$.
- If $\varphi, \psi \in D$, we say $\varphi =_C \psi$ if $C \subseteq^* \{i : \varphi(i)\downarrow = \psi(i)\downarrow\}$.
- Denote the $=_C$-equivalence class of $\varphi \in D$ by $[\varphi]$.

"φ and ψ are the same element of the cohesive power if C says that they are equal as functions."

Satisfaction in Cohesive Powers

Fix a countable language $\mathcal{L}$, computable $\mathcal{L}$-structure $\mathcal{A}$ and cohesive set C. We define the *cohesive power of* $\mathcal{A}$ *over* C, $\prod_C \mathcal{A}$, by

- $|\prod_C \mathcal{A}| = \{[\varphi] : C \subseteq^* \mathrm{dom}(\varphi)\}$.
- For n-ary relation $R \in \mathcal{L}$, let $\prod_C \mathcal{A} \models R([\varphi_1], \ldots, [\varphi_n])$ if and only if

$$C \subseteq^* \left\{ i : i \in \bigcap_{j \leq n} \mathrm{dom}(\varphi_j) \text{ and } \mathcal{A} \models R(\varphi_1(i), \ldots, \varphi_n(i)) \right\}.$$

- For n-ary function symbol f and $\varphi_1, \ldots, \varphi_n$, let

$$\psi(i) = \begin{cases} f^{\mathcal{A}}(\varphi_0(i), \ldots, \varphi_n(i)) & \text{if } i \in \bigcap \mathrm{dom}_{j \leq n}(\varphi_j) \\ \uparrow & \text{otherwise.} \end{cases}.$$

  Then, define $f^{\prod_C \mathcal{A}}([\varphi_1], \ldots, [\varphi_n]) = [\psi]$.

Canonical Embedding of $\mathcal{A}$ Into $\prod_C \mathcal{A}$

For constant symbol c, let $\psi(i) = c^{\mathcal{A}}$ for all i. Then, we interpret

$$\prod_C \mathcal{A} \models c = [\psi].$$

We can do this for each element of $\mathcal{A}$.

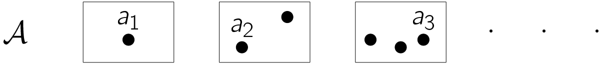
Let $a \in \mathcal{A}$. Define $f_a(i) = a$ for all i. Then, the map $a \mapsto [f_a]$ embeds $\mathcal{A}$ into $\prod_C \mathcal{A}$.

So $\prod_C \mathcal{A}$ contains a copy of $\mathcal{A}$.

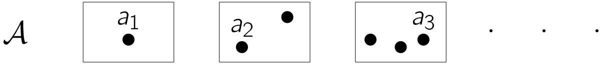
Let K be a finite set and φ partial computable such that $C \subseteq^* \{i : \varphi(i) \downarrow \in K\}$. Then, there is $k \in K$ such that $C \subseteq^* \{i : \varphi(i) \downarrow = k\}$ and hence $[\varphi] = [f_k]$, so $[\varphi]$ is part of the canonical embedding. Lesson: **functions with finite ranges on C are constant on C.**

Example: Equivalence Relation

- Cohesive powers of ω can be complicated and depend on the cohesive set C.
- Cohesive powers of equivalence relations are relatively simpler.
- Let $\mathcal{L} = \{\sim\}$ be the language of equivalence relations and let $\mathcal{A}$ be a computable structure with exactly one equivalence class of each finite size $n \in \omega$, each represented by some $a_n \in \omega$.
- Fix cohesive set C.
- We will show that $\prod_C \mathcal{A}$ has exactly one equivalence class of each finite size and countably many countable equivalence classes, completely specifying its isomorphism type.



Example: Equivalence Relation



- Fix partial computable function φ with $C \subseteq^* \mathrm{dom}(\varphi)$. Consider $B_n = \{i : \varphi(i) \sim_{\mathcal{A}} a_j \text{ for some } j < n\}$.
- Suppose $C \subseteq^* B_n$ for some n.
  - Then, φ has finite range on C and is therefore constant on C. $[\varphi]$ is in a class of size at most n.
- Suppose $C \not\subseteq^* B_n$ for each n.
  - Then, $[\varphi] \neq [f_a]$ for any $a \in \mathcal{A}$, and furthermore, $\prod_C \mathcal{A} \not\models [\varphi] \sim [f_a]$ for any $a \in \mathcal{A}$.
    - This shows that each $[f_a]$ is in a $\sim_{\prod_C \mathcal{A}}$-equivalence class of the same size as the $\sim_{\mathcal{A}}$-equivalence class of a.
  - To complete our description of the isomorphism type of $\prod_C \mathcal{A}$, we only need to show that $[\varphi]$ is part of an infinite equivalence class.

Example: Equivalence Relation

- We construct partial computable $\varphi_1, \varphi_2, \ldots$ such that $\prod_C \mathcal{A} \models [\varphi_n] \sim [\varphi_m] \sim \varphi$ and $[\varphi] \neq [\varphi_n] \neq [\varphi_m]$ for each $n \neq m$.
- Idea: set $\varphi_n(i)$ equal to the n'th element of the $\sim_{\mathcal{A}}$-equivalence class $[\varphi(i)]_{\sim_{\mathcal{A}}}$ of $\varphi(i)$.
- We cannot compute the size of equivalence classes. However, we can give them lower bounds. Define

$$\varphi_n(i) = \begin{cases} \text{The } n\text{'th element of } [\varphi(i)]_{\sim_{\mathcal{A}}} & \text{if } \varphi(i)\downarrow \text{ and we find} \\ & \text{at least } n \text{ elements of } [\varphi(i)]_{\sim_{\mathcal{A}}} \\ \uparrow & \text{otherwise} \end{cases}.$$

Then, each φ_n is pointwise $\sim_{\mathcal{A}}$-equivalent to φ and eventually pointwise non-equal to φ.

Fundamental Theorem of Cohesive Powers

Fundamental Theorem of Cohesive Powers

Cohesive powers satisfy the following analogue to Łoś's theorem.

Theorem (Dimitrov, 2009)

Let $\mathcal{A}$ be a computable $\mathcal{L}$-structure and C be a cohesive set. Then,

1. *Let $\Psi(x_1, \ldots, x_m)$ be a Δ_2 formula. Then, for any $[\varphi_1], \ldots, [\varphi_m] \in \prod_C \mathcal{A}$,*

$$\prod_C \mathcal{A} \models \Phi([\varphi_1], \ldots, [\varphi_m]) \iff$$

$$C \subseteq^* \left\{ i : i \in \bigcap_{j \leq m} dom(\varphi_j) \text{ and } \mathcal{A} \models \Phi(\varphi_1(i) \ldots, \varphi_m(i)) \right\}.$$

2. *Let Φ be a Σ_3 $\mathcal{L}$-sentence. Then $\mathcal{A} \models \Phi$ implies $\prod_C \mathcal{A} \models \Phi$.*
3. *Let Φ is a Δ_3 $\mathcal{L}$-sentence. Then, $\prod_C \mathcal{A} \models \Phi$ if and only if $\mathcal{A} \models \Phi$.*

Applications of the Fundamental Theorem

Example (Cohesive Powers of Equivalence Relations are Equivalence Relations)

The axioms of equivalence relations are

- Reflexivity: $(\forall x)(x \sim x)$ is Π_1.
- Symmetry: $(\forall x \forall y)(x \sim y \leftrightarrow y \sim x)$ is Π_1.
- Transitivity: $\forall x \forall y \forall z((x \sim y \wedge y \sim z) \rightarrow x \sim z)$ is Π_1.

Since all of these axioms are Σ_3, we have that cohesive powers of equivalence relations are equivalence relations.

Example

An element x being in an equivalence class of size less than 3 is expressible by

$$(\forall y_1, y_2)(x \sim y_1 \sim y_2 \rightarrow (x = y_1 \vee x = y_2 \vee y_1 = y_2)),$$

a Π_1 formula. Therefore, this property is preserved by the cohesive power.

Applications of the Fundamental Theorem

Example

There existing exactly one equivalence class of size n is also Σ_2, so this is also preserved by the cohesive power.

Example

The existence of an infinite equivalence class is not expressible in the language of equivalence relations. Hence it is not preserved by cohesive powers.

Example

x being contained in an infinite equivalence class is a expressible by a family of Σ_1 sentences. Hence, this property is preserved by cohesive powers.

Refinement of the Fundamental Theorem

Theorem (Dimitrov, Harizanov, Morozov, Shafer, Soskova, and Vatev, 2022)

Suppose that the Σ_n diagram of $\mathcal{A}$ is decidable. Then

1. *Let $\Psi(x_1, \ldots, x_m)$ be a Δ_{n+2} formula. Then, for any $[\varphi_1], \ldots, [\varphi_m] \in \prod_C \mathcal{A}$,*

$$\prod_C \mathcal{A} \models \Phi([\varphi_1], \ldots, [\varphi_m]) \iff$$

$$C \subseteq^* \left\{ i : i \in \bigcap_{j \leq m} dom(\varphi_j) \text{ and } \mathcal{A} \models \Phi(\varphi_1(i) \ldots, \varphi_m(i)) \right\}.$$

2. *Let Φ be a Σ_{n+3} $\mathcal{L}$-sentence. Then $\mathcal{A} \models \Phi$ implies $\prod_C \mathcal{A} \models \Phi$.*
3. *Let Φ is a Δ_{n+3} $\mathcal{L}$-sentence. Then, $\prod_C \mathcal{A} \models \Phi$ if and only if $\mathcal{A} \models \Phi$.*

Relativization of the Fundamental Theorem

We can also gain more conservation be increasing the amount of cohesiveness and expanding our domain.

Theorem

Let A be a computable $\mathcal{L}$-structure and C be a Σ_n-cohesive set. Let $\mathcal{B}$ be the cohesive power of A over C, modifying the construction to include partial $\emptyset^{(n-1)}$-computable functions in the domain. Then,

1. *If Φ is a Σ_{n+2} $\mathcal{L}$-sentence then $\mathcal{M} \models \Phi$ implies $\mathcal{B} \models \Phi$.*
2. *If Φ is a Δ_{n+2} $\mathcal{L}$-sentence then $\mathcal{B} \models \Phi$ if and only if $\mathcal{M} \models \Phi$.*

Categorizations

Equivalence Relations

Theorem

Let $\mathcal{A}$ be a computable equivalence relation and let C be any cohesive set. Then,

1. *$\mathcal{A}$ and $\prod_C \mathcal{A}$ have the exact same number or equivalence classes of each finite size.*
2. *$\prod_C \mathcal{A}$ has an infinite equivalence class if and only if, for each N, $\mathcal{A}$ has an equivalence class of size larger than N.*
3. *$\prod_C \mathcal{A}$ has infinitely many infinite equivalence classes if and only if $\mathcal{A}$ has an infinite set of equivalence classes of unbounded sizes.*

(1) follows from the fundamental theorem of cohesive powers. (2) has two cases. In case that $\mathcal{A}$ has only finite equivalence classes, the previous construction still holds. In the case that $\mathcal{A}$ has an infinite equivalence class, apply the fundamental theorem of cohesive powers. (3) follows from the previous construction as well as (1) and (2).

Linear Orders

Fix the language $\mathcal{L} = \{\leq\}$. Dimitrov, Harizanov, Morozov, Shafer, Soskova, and Vatev (2022) study the cohesive powers of linear orders. The axioms of (dense) linear orders (without endpoints) are Π_1 (Π_2), so cohesive powers of (dense) linear orders (without endpoints) are also (dense) linear orders (without endpoints).

Furthermore, certain operations on linear orders are preserved by cohesive powers.

Theorem

Fix linear orders $\mathcal{A}_1$ and $\mathcal{A}_2$. Let $\mathcal{A}^*$ denote the reversal of $\mathcal{A}$. Then,

- $\prod_C \mathcal{A}_1 + \mathcal{A}_2 = \prod_C \mathcal{A}_1 + \prod_C \mathcal{A}_1$.
- $\prod_C (\mathcal{A}_1 \mathcal{A}_2) = \left(\prod_C \mathcal{A}_1\right)\left(\prod_C \mathcal{A}_2\right)$
- $\prod_C \mathcal{A}^* = \left(\prod_C \mathcal{A}\right)^*$

Cohesive powers of $(\omega, \leq)$

Denote the order type of $\mathbb{N}$ by ω, the order type of $\mathbb{Z}$ by ζ and the order type of $\mathbb{Q}$ by η. Fix computable copy $\mathcal{A}$ of ω with computable successor function.

Theorem

For any cohesive set C, the cohesive power $\prod_C \mathcal{A}$ has order type $\omega + \zeta\eta$ (i.e., an initial segment of naturals followed by dense copies of ζ).

Theorem

There are computable copies of ω with non-computable successor functions whose cohesive powers are still isomorphic to $\omega + \zeta\eta$.

Theorem

There are also computable copies of ω with a cohesive power with order type $\omega + \eta$.

Cohesive Powers of $(\omega, \leq)$ (cont.)

Theorem

Let $k_0, \ldots, k_n$ be nonzero natural numbers. There is a computable copy $\mathcal{A}$ of ω with a cohesive power that has order type ω+densely many copies of $k_0, \ldots, k_n$.

So cohesive powers of ω can go from being very simple to being very complicated. These constructions all hinge upon manipulating the successor function of $\mathcal{A}$ in a way specific to C.

Let $\mathcal{N}$ be the standard presentation of the natural numbers ordered under $\leq_{\mathbb{N}}$. We will demonstrate the following theorem.

Theorem

For any cohesive set $\mathbb{C}$, the cohesive power $\prod_C \mathbb{N} \cong \omega + \zeta\eta$.

Lemma

The image of $\mathbb{N}$ in the canonical embedding of $\mathbb{N}$ into $\prod_C \mathbb{N}$ is an initial segment of order type ω.

Proof.

Suppose $\prod_C \mathbb{N} \models [\varphi] \leq f_k$ for $k \in \mathbb{N}$. Then, $\varphi(i) \leq k$ for all i, so φ has finite range. By cohesiveness of C, we have that $[\varphi] = [f_j]$ for some $j \leq k$. □

Say that $[\varphi] \in \prod_C \mathbb{N}$ is nonstandard if $\prod_C \mathbb{N} \models [\varphi] \geq f_k$ for all $k \in \mathbb{N}$.

Lemma

Then, $[\varphi]$ is nonstandard if and only if $\lim_{i \in C} \varphi(i) = \infty$.

Proof.

($\Rightarrow$): If φ is standard then we have already seen that φ is eventually constant on C.
($\Leftarrow$): If $\lim_{i \in C} \varphi(i) = \infty$ then $C \subseteq^* \{i : \varphi(i) \geq f_k(i)\}$ for each k. Thus, $\prod_C \mathbb{N} \models [\varphi] \geq f_k$ for all $k \in \mathbb{N}$. □

Lemma

Let $[\varphi]$ *be nonstandard. Then there are nonstandard* $[\psi^+]$ *and* $[\psi^-]$ *such that the intervals* $([\psi^-], [\varphi])$ *and* $([\varphi], [\psi^+])$ *of* $\prod_C \mathbb{N}$ *are both infinite.*

Proof.

Define

$$\psi^-(i) = \begin{cases} \left\lfloor \frac{\varphi(i)}{2} \right\rfloor & \text{if } \varphi(i)\downarrow \\ \uparrow & \text{otherwise} \end{cases}.$$

We can definite infinitely many functions between ψ^- and φ by adjusting the fraction. We find ψ^+ similarly. □

Lemma

Each nonstandard $[\varphi]$ *is contained in a copy of* ζ.

Proof.

$[\varphi+1]$ is the successor of $[\varphi]$ and $[\varphi-1]$ is the predecessor of φ because being adjacent is a Π_1 property. □

Injection Structures

- Cenzer, Harizanov, and Remmel (2014) study computability theoretic properties of injection structures, which are structures with a single injective function.
- Injection structures seem like a natural counterpart to linear orders with non-computable successors.
- Orbits of injection structures form a natural equivalence relation.
- Injection structures can be decomposed into finite cycles, $\mathbb{Z}$-chains and ω-chains.

Injection Structures

Theorem

Let $\mathcal{A}$ be a computable injection structure and let C be a cohesive set. Then

- *$\prod_C \mathcal{A}$ and $\mathcal{A}$ have exactly the same number of ω-chains and finite cycles of each size $n \in \mathbb{N}$.*
- *$\prod_C \mathcal{A}$ either has infinitely many $\mathbb{Z}$-chains or it has zero $\mathbb{Z}$-chains.*

Furthermore, $\prod_C \mathcal{A}$ has zero $\mathbb{Z}$-chains if and only if $\mathcal{A}$ consists entirely of finite cycles whose sizes are all below some upper bound $N \in \omega$.

Proof:

- Having k many elements outside the range of f is Σ_2, so is preserved by cohesive powers.
- Having k many cycles of size n is also Σ_2, so is preserved by cohesive powers.

It remains to discuss when the cohesive power has $\mathbb{Z}$-chains.

Constructing $\mathbb{Z}$-chains

What do we need to construct a partial computable function that is part of a $\mathbb{Z}$-chain in $\mathcal{A}$?

- Let $a_1, a_2, \ldots$ be elements of $\mathcal{A}$ such that $f^{2i}(a_i) \neq f^j(a_i)$ for all $0 \leq j \leq 2i$.
- Define partial function $\varphi(i) = f^i(a_i)$.
- For each n, the set $\{i : (\exists j < n) f^n(\varphi(i)) = f^j(\varphi(i))\}$ is finite, so $[\varphi]$ has infinitely many successors.
- We can also take the n'th predecessor of φ co-finitely, so $[\varphi]$ has infinitely many predecessors.
- We can build infinitely many such φ that are cofinitely pointwise non-equal and never in the same orbit by varying the ratio between i and the number of times we apply f to a_i.

So all we need to build infinitely many $\mathbb{Z}$-chains is to compute such a sequence of $a_1, a_2, \ldots$.

Computing $a_1, a_2, \ldots$

- Need $a_1, a_2, \ldots$ such that $f^{2i}(a_i) \neq f^j(a_i)$ for all $0 \leq j \leq 2i$.
- We can compute such a sequence if
  - Either $\mathcal{A}$ has a single infinite orbit or
  - $\mathcal{A}$ has unbounded finite orbits.

On the other hand, a bound on the size of the orbits is expressible in a Π_1 way, so it would be preserved by cohesive power.

Questions

- 2 to 1 function structures.
- Structures with more complicated definable relations/functions?
  - Might use marker extension to do so

Thank you!

Have a great evening!

References I

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